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On the Stability of Results based upon Average Calculations, considered with reference to the Number of Transactions embraced.

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AN article published in this *Journal* last January (vol. viii., p. 316), proposed itself the problem of furnishing a test whether the degree of uniformity, or the reverse, observed in a table of statistics, required to be assigned to any other cause than the relative magnitude of the numbers of combinations which each figure in the table represented. The formulæ obtained in that article gave a general solution to the problem proposed, by giving an expression for the ratios of the numbers of such combinations which would be embraced under any figure in the table; and, consequently, expressing the relative frequency with which each number should be expected to occur from the mere consideration of counting the combinations represented by it.

These formulæ will aid in the solution of another and a much more useful problem—namely, to give us the power of prediction, in some very important practical cases, as to the *stability* to be expected in future results calculated from the observation of an average. The simplest case is that where the data on which the calculation is founded are taken from so large sources as to be considered complete.

The problem proposed in the following pages is this: *To show the degree of stability to be expected in the numbers representing the deaths in a given period to be experienced in the course of a life assurance business of given amount*—the problem, for the present, being treated on the supposition that the data on which the calculations of the Office are based are sufficiently large to furnish the true average.

The relation, it may be remarked, between the extensiveness of the data and the accuracy of the average deduced from them, is implicitly given by the formulæ in page 321 of the above article, which may be the subject of future expansion. Enough, for the present, to observe, that when a (the number representing the number of years for which observations are taken) is very large, the formulæ approximate to expressions independent of a altogether—which, looking at the considerations on which these formulæ are obtained, means to say that the true average has been approximately arrived at. Again: the true average being arrived at, the formulæ on page 324 represent the relative frequency with which numbers differing from the average by any given amount may be expected to occur; but as these formulæ may be obtained more simply and from less complicated considerations than those involved in the above article, I shall proceed to obtain them by a more direct method.

Suppose that we have a sufficient number of observations for obtaining with certainty the *average* percentage of persons of a given class to whom a certain event happens in a given time.

Let a society consist of n persons, who are supposed so far unconnected with each other that the event contemplated happening to one does not make it more or less likely to happen to another, and that of these n persons we know of no reason why it should happen to one rather than another, but that they are all of the given class on which the above observations are made. And suppose that, according to the average above found, the number out of n persons to which the event should happen is b.

Let us designate the above persons by $A_1, A_2 \ldots A_n$; then, by the above hypothesis, the probability of the event happening to A_1 is

$$\frac{b}{n},$$

and the probability of its *not* happening to A_2 is

$$1-\frac{b}{n}.$$

The probability, therefore, of its happening to A_1 and not to A_2 is

$$\frac{b}{n}\left(1-\frac{b}{n}\right);$$

and that of its happening to A_1 and neither to A_2, A_3, nor A_n, is

$$\frac{b}{n}\left(1-\frac{b}{n}\right)^{n-1}.$$

But, of course, the same will be the probability of its happening to A_2 and not to A_1, A_3, nor A_n; and, therefore, the probability of its happening to one person only out of the whole society is

$$n.\frac{b}{n}\left(1-\frac{b}{n}\right)^{n-1}=b.\left(1-\frac{b}{n}\right)^{n-1}.$$

The probability of its happening to both A_1 and A_2 is

$$\left(\frac{b}{n}\right)^2,$$

and that of its happening to both A_1 and A_2, and not to any of the rest,

$$\left(\frac{b}{n}\right)^2\left(1-\frac{b}{n}\right)^{n-2}.$$

The probability, therefore, of its happening to *two persons only* out of the society is

$$\frac{n(n-1)}{2}\cdot\left(\frac{b}{n}\right)^2\left(1-\frac{b}{n}\right)^{n-2};$$

that of its happening to *three persons only,*

$$\frac{n(n-1)(n-2)}{1.2.3}\cdot\left(\frac{b}{n}\right)^3\left(1-\frac{b}{n}\right)^{n-3}.$$

The law is evident, and we shall only further write down the probability of its happening to exactly b persons out of the Society, and no more nor less, which is

$$\frac{n(n-1)(n-2)\ldots(n-b+1)}{1.2.3\ldots\ldots b}\left(\frac{b}{n}\right)^b\left(1-\frac{b}{n}\right)^{n-b}.^*$$

This will be the *most probable number,* for it has been obtained from the last by multiplying by

$$\frac{n-b+1}{b}\cdot\frac{b}{n}\cdot\left(1-\frac{b}{n}\right)^{-1}=\frac{n-b+1}{b}\cdot\frac{b}{n-b}=\frac{n-b+1}{n-b},\text{ which is} > 1;$$

and the next would be obtained from it by multiplying by

$$\frac{n-b}{b+1}\cdot\frac{b}{n}\cdot\left(1-\frac{b}{n}\right)^{-1}=\frac{n-b}{b+1}\cdot\frac{b}{n-b}=\frac{b}{b+1},\text{ which is} < 1.$$

If we now compare the probability of the number b being the exact number out of the society to whom the event happens, with that of $b-1$, $b-2$, and so on being the number, and then with that of $b+1$, $b+2$, and so on being the number, the ratios will be—

---

* So far, this investigation is substantially identical with one which is well known—the language of the above being, for the sake of clearness, adapted to the case here contemplated. The simplifications which follow, admitting the definite working out of the problem in the most useful practical form, are believed to be new.

The mathematical demonstration of the stability of average results, a principle on the practical truth of which the whole fabric of assurance business rests, has, I believe, hitherto rested on very general grounds borrowed from investigations of La Place, which were chiefly worked out in a totally different direction, and apply only to the problem now under consideration in the case where the numbers are very large. In other words, it has been proved that the results of extensive transactions of this kind approach to stability, but no definite relation has hitherto been given between the extent of transactions and the degree of stability to be expected.

Probability of b being the number to that of $(b-1)$

$$=\frac{n-b+1}{b}\cdot\frac{b}{n-b} \quad \ldots\ldots\ldots\ldots \quad a_1$$

Probability of b being the number to that of $(b-2)$

$$=\frac{(n-b+1)(n-b+2)}{b\quad(b-1)}\cdot\left(\frac{b}{n-b}\right)^2 \quad \ldots\ldots \quad a_2$$

Probability of b being the number to that of $(b-3)$

$$=\frac{(n-b+1)(n-b+2)(n-b+3)}{b\quad(b-1)\quad(b-2)}\cdot\left(\frac{b}{n-b}\right)^3 \quad . \quad a_3$$

and so on; and

Probability of b being the number to that of $(b+1)$

$$=\frac{b+1}{n-b}\cdot\frac{n-b}{b} \quad \ldots\ldots\ldots\ldots \quad a_1$$

Probability of b being the number to that of $(b+2)$

$$=\frac{(b+1)(b+2)}{(n-b)(n-b-1)}\cdot\left(\frac{n-b}{b}\right)^2 \quad \ldots\ldots \quad a_2$$

Probability of b being the number to that of $(b+3)$

$$=\frac{(b+1)\,(b+2)\,(b+3)}{(n-b)\,(n-b-1)(n-b-2)}\cdot\left(\frac{n-b}{b}\right)^3 \quad . . \quad a_3$$

and so on.

It is evident that, after getting a certain length, these numbers will increase with enormous rapidity. That rapidity of increase is the test of the stability of calculations proceeding upon average results, and we shall best show how it may be applied to practical cases, by dealing with actual numbers and taking advantage of all the approximate simplifications which they will supply to us.

It is plain that if n is very large compared to b, we may neglect such factors as $\frac{n-b+1}{n-b}$, $\frac{n-b+2}{n-b}$, $\frac{n-b}{n-b-1}$; in which case the above ratios will be very much simplified, and will *depend on the value of b alone.* They will become

$$a_1=\frac{b}{b}=1, \qquad a_1=\frac{b+1}{b},$$

$$a_2=\frac{b}{b-1}, \qquad a_2=\frac{(b+1)(b+2)}{b^2},$$

$$a_3=\frac{b^2}{(b-1)(b-2)}, \qquad a_3=\frac{(b+1)(b+2)(b+3)}{b^3},$$

and so on.

The practical extent to which we may extend this simplification will soon appear from a numerical example. Suppose $n=1{,}000$, and let $b=7$, which will very nearly represent the mortality among persons of 20 years of age according to the Carlisle Tables; we have then $n-b=993$, and, substituting these values in the expression first given for a_1, a_a, &c., we have—

$$a_1=\frac{994}{7}\cdot\frac{7}{993}=\frac{994}{993}\times 1,$$

$$a_2=\frac{994.995}{7\;.\;6}\times\left(\frac{7}{993}\right)^2=\frac{994}{993}\cdot\frac{995}{993}\times\frac{7}{6},$$

$$a_3=\frac{994.995.996}{7\;.\;6\;.\;5}\cdot\left(\frac{7}{993}\right)^3=\frac{994}{993}\cdot\frac{995}{993}\cdot\frac{996}{993}\times\frac{7}{6}\cdot\frac{7}{5},$$

and so on; and

$$\alpha_1=\frac{8}{993}\cdot\frac{993}{7}=\frac{8}{7},$$

$$\alpha_2=\frac{8\;.\;9}{993.992}\cdot\left(\frac{993}{7}\right)^2=\frac{8}{7}\cdot\frac{9}{7}\cdot\frac{993}{992},$$

$$\alpha_3=\frac{8\;.\;9\;.10}{993.992.991}\cdot\left(\frac{993}{7}\right)^3=\frac{8}{7}\cdot\frac{9}{9}\cdot\frac{10}{7}\times\frac{993}{992}\cdot\frac{993}{991},$$

and so on.

Now, it will not make much difference in the rapidity with which these ratios, a_1, a_2, a_3, &c., and α_1, α_2, α_3, increase, *especially when we get beyond the first two or three,* if we neglect such a fraction as $\frac{996}{993}$ in comparison with $\frac{7}{5}$, or $\frac{993}{991}$ in comparison with $\frac{10}{7}$. We need only remember, that if we had not neglected these fractions the stability of results would have been found to be rather greater.

Again: let $n=1{,}000$, as before, and let $b=96$, which will nearly represent the mortality at the age of 75 from the same tables; then $n-b=904$, and we have—

$$a_1=\frac{905}{96}\cdot\frac{96}{904}=\frac{905}{904}\times 1,$$

$$a_2=\frac{905.906}{96.95}\left(\frac{96}{904}\right)^2=\frac{905}{904}\cdot\frac{906}{904}\times\frac{96}{95},$$

$$a_3=\frac{905.906.907}{96.95.94}\left(\frac{96}{904}\right)^3=\frac{905}{904}\cdot\frac{906}{904}\cdot\frac{907}{904}\times\frac{96}{95}\cdot\frac{96}{94},$$

and

$$\alpha_1=\frac{97}{904}\cdot\frac{904}{96}=\frac{97}{97},$$

$$\alpha_2=\frac{97.98}{904.903}\cdot\left(\frac{904}{96}\right)^2=\frac{97}{96}\cdot\frac{98}{96}\times\frac{904}{903},$$

$$\alpha_3=\frac{97.98.99}{904.903.902}\cdot\left(\frac{904}{96}\right)^3=\frac{97}{96}\cdot\frac{98}{96}\cdot\frac{99}{96}\times\frac{904}{903}\cdot\frac{904}{902}.$$

Now, as before, and having an eye to the use we are about to make of these figures, it will not materially affect our results if we neglect the fraction involving the large figures in comparison of the others corresponding to them. We might, perhaps, retain all through a_1, a_2, a_3, the factor $\frac{905}{904}$; but even this may, for the present, be neglected—only remembering that in what follows we shall slightly underestimate the stability of results to be expected from considerations based on an average.

Now, suppose the lives assured in an Office comprise various ages between 20 and 75 (if there are a few over the latter age, which we fixed on merely for illustration, it will not materially affect our results); and suppose that, according to the tables, taking the due proportion of lives to each age, the whole number of lives that should fall next year is 96. It scarcely requires proof that what I call the stability of this expectation—that is, the rapidity of increase of the numbers representing the ratio of the probability of this result to the respective results of 95, 94, 93, &c., 97, 98, 99, &c., lives falling—will be intermediate to the stability of expected results in two hypothetical cases—namely, these:—

Take first the case of an Office which should have nothing but lives aged 75 assured, and whose expected number of deaths next year was 96.

Then take the case of an Office which should have nothing but lives aged 20 assured, and whose expected number of deaths next year was also 96.

The first hypothetical Office must have just 1,000 lives assured, and the values of a_1, a_2, &c., in its case, are just those found above on p. 220.

The second hypothetical Office must have a number of lives assured equal to $96 \times \frac{1,000}{7} = 13,914$. We should then have to make $n = 13,914$, $b = 96$, and $n - b = 13,818$.

But from what has been shown above, in estimating the stability of either, we might have, for all practical purposes, neglected the value of n. The stability in either case, therefore, is nearly the same—depends on $b = 96$ alone, and not on n—and may be estimated from the values of a_1, a_2, &c., on p. 219; and the stability thus estimated, being a sufficient approximation to the stability of results to be expected in either of the hypothetical cases, is a sufficient approximation to that in the real case which lies between them.

The formulæ on p. 219, then, easily supply the means of constructing tables by which it may be shown at a glance, in the case of an Office expecting (according to average calculations) that a certain number (say 50) of lives will fall in the course of the year, what are the relative probabilities of the number of lives actually falling being 50, 49, 48, &c., 51, 52, 53, &c.

This being done, it will easily appear how, finally, we can form a table showing, if the number on an average calculation be given, what is the probability of the true number being found to differ from the calculated one by more than a given percentage of its amount, which I conceive to be the best form in which the relation between the stability of an assurance business and the number of its transactions can be exhibited.

After what has been said, the tables which follow will sufficiently explain themselves. In what follows we shall adopt the following notation:—

$$\phi_1(b)=\frac{b}{b}=1,$$

$$\phi_2(b)=\frac{b-1}{b},$$

$$\phi_3(b)=\frac{(b-1)(b-2)}{b^2},$$

$$\cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot$$

$$\phi_n(b)=\frac{(b-1)(b-2)\ .\ .\ .\ .\ (b-n+1)}{b^{n-1}},$$

and

$$\psi_1(b)=\frac{b}{b+1},$$

$$\psi_2(b)=\frac{b^2}{(b+1)(b+2)},$$

$$\psi_3(b)=\frac{b^3}{(b+1)(b+2)(b+3)},$$

$$\cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot$$

$$\psi_n(b)=\frac{b^n}{(b+1)(b+2)\ .\ .\ .\ .\ (b+n)},$$

the reciprocals of the expressions on p. 219.

TABLE I.—*Preparatory Division Table, by Logarithms, giving the value of log.* $\frac{b+x}{b}$ *or* $\frac{b}{b-x}$, *for all values of b from* 1 *to* 20 *and x from* 1 *to* 20.

Divisors (x)	Dividend (b) 1.	2.	3.	4.	5.	6.	7.	8.	9.	10.
1	·0000000	·3010300	·4771213	·6020600	·6989700	·7781513	·8450980	·9030900	·9542425	1·0000000
2		·0000000	·1760913	·3010300	·3979400	·4771213	·5440680	·6020600	·6532125	·6989700
3			·0000000	·1249387	·2218487	·3010300	·3679767	·4259687	·4771212	·5228787
4				·0000000	·0969100	·1760913	·2430380	·3010300	·3521825	·3979400
5					·0000000	·0791813	·1461280	·2041200	·2552726	·3010300
6						·0000000	·0669467	·1249387	·1760912	·2218487
7							·0000000	·0579920	·1091445	·1549020
8								·0000000	·0511525	·0969100
9									·0000000	·0457575

x=	b=11.	b=12.	b=13.	b=14.	b=15.	b=16.	b=17.	b=18.	b=19.	, b=20.
1	1·0413927	1·0791812	1·1139434	1·1461280	1·1760913	1·2041200	1·2304489	1·2552725	1·2787536	1·3010300
2	·7403627	·7781512	·8129134	·8450980	·8750613	·9030900	·9294189	·9542425	·9777236	1·0000000
3	·5642714	·6020599	·6368221	·6690067	·6989700	·7269987	·7533276	·7781512	·8016323	·8239087
4	·4393327	·4771212	·5118834	·5440680	·5740313	·6020600	·6283889	·6532125	·6766936	·6989700
5	·3424227	·3802112	·4149734	·4471580	·4771213	·5051500	·5314789	·5565025	·5797836	·6020600
6	·2632414	·3010300	·3357921	·3679767	·3979400	·4259687	·4522976	·4771212	·5006023	·5228787
7	·1962947	·2340832	·2688454	·3010300	·3309933	·3590220	·3853509	·4101745	·4336556	·4559320
8	·1283027	·1760912	·2108534	·2430380	·2730013	·3010300	·3273589	·3521825	·3756636	·3979400
9	·0871502	·1249387	·1597009	·1908855	·2218488	·2498775	·2762064	·3010300	·3245111	·3467875
10	·0413927	·0791812	·1139434	·1461280	·1760913	·2041200	·2304489	·2552725	·2787536	·3010300
11	·0000000	·0377885	·0725507	·1047353	·1346986	·1627273	·1890562	·2138798	·2373609	·2596373
12		·0000000	·0347622	·0669468	·0969101	·1249388	·1512677	·1760913	·1995724	·2218488
13			·0000000	·0321846	·0621479	·0901766	·1165055	·1413291	·1648102	·1870866
14				·0000000	·0299633	·0579920	·0843209	·1091445	·1326256	·1549020
15					·0000000	·0280287	·0543576	·0791812	·1026623	·1249387
16						·0000000	·0263289	·0511525	·0746336	·0969100
17							·0000000	·0248236	·0483047	·0705811
18								·0000000	·0234811	·0457575
19									·0000000	·0222764
20										·0000000

It is evident that, commencing at the bottom of a column corresponding to any given value of b, and adding vertically upwards, the numbers produced will be the logs. of the quantities $\frac{1}{\phi_1(b)}, \frac{1}{\phi_2(b)}, \frac{1}{\phi_3(b)}$, and so on; and adding horizontally, we shall, in the same way, produce the logs. of $\frac{1}{\psi_1(b)}, \frac{1}{\psi_2(b)}, \frac{1}{\psi_3(b)}$, and so on.

TABLE I. (*continued*).

$x=$	· $b=21.$	$b=22.$	$b=23.$	$b=24.$	$b=25.$	$b=26.$	$b=27.$	$b=28.$	$b=29.$	$b=30.$	$b=31.$
1	1·3222193	1·3424227	1·3617278	1·3802112	1·3979400	1·4149733	1·4313638	1·4471580	1·4623980	1·4771213	1·4913617
2	1·0211893	1·0413927									
3	·8450980	·8653014	·8846065								
4	·7201593	·7403627	·7596678	·7781513							
5	·6232493	·6434527	·6627578	·6812412	·6989700						
6	·5440680	·5642714	·5835765	·6020600	·6197887	·6368221					
7	·4771213	·4973247	·5166298	·5351132	·5528420	·5698754	·5862658				
8	·4191293	·4393327	·4486378	·4771213	·4948500	·5118834	·5282738	·5440680			
9	·3679768	·3881802	·4074853	·4259687	·4436975	·4607309	.4771213	·4929155	·5081555		
10	·3222193	·3424227	·3617278	·3802112	·3979400	·4149734	·4313638	·4471580	·4623980	·4771213	
11	·2808266	·3010300	·3203351	·3388185	·3565473	·3735806	·3899711	·4057653	·4210053	·4357286	·4499690
12	·2430381	·2632414	·2825466	·3010300	·3187588	·3357921	·3521826	·3679767	·3832168	·3979410	·4121805
13	·2082759	·2284793	·2477844	·2662678	·2839966	·3010300	·3174204	·3332146	·3484546	·3631779	·3774183
14	·1760913	·1962947	·2155998	·2340832	·2518120	·2688454	·2852358	·3010300	·3162700	·3309933	·3452337
15	·1461280	·1663314	·1856365	·2041199	·2218487	·2388820	·2552725	·2710667	·2863067	·3010300	·3152704
16	·1180993	·1283027	·1576078	·1760912	·1938200	·2108534	·2272438	·2430380	·2582780	·2730013	·2872417
17	·0917704	·1129738	·1312789	·1497623	·1674911	·1845244	·2009149	·2167091	·2319491	·2466724	·2609128
18	·0669468	·0871502	·1064553	·1249387	·1426675	·1597009	·1760913	·1908855	·2071255	·2218488	·2360892
19	·0434657	·0636691	·0829742	·1014576	·1191864	·1362197	·1526102	·1684044	·1836444	·1983677	·2126081
20	·0211893	·0413927	·0606978	·0791812	·0969100	·1139434	·1303338	·1461280	·1613680	·1760913	·1903317

$x=$	$b=32.$	$b=33.$	$b=34.$	$b=35.$	$b=36.$	$b=37.$	$b=38.$	$b=39.$	$b=40.$	$b=41.$	$b=42.$
	1·5051500	1·5185139	1·5314789	1·5440680	1·5563025	1·5682017	1·5797836	1·5910646	1·6020600		
12	·4259687										
13	·3912066	·4045705									
14	·3590220	·3723859	·3853509								
15	·3290587	·3424226	·3553876	·3679767							
16	·3010300	·3143939	·3273589	·3399480	·3521825						
17	·2747011	·2880650	·3010300	·3136191	·3258536	·3377528	·3493347				
18	·2498775	·2632414	·2762064	·2887955	·3010300	·3129292	·3245111	·3357921			
19	·2263964	·2397593	·2527253	·2653144	·2775489	·2894481	·3010300	·3123110	·3233064	·3340303	
20	·2041200	·2174839	·2304489	·2430380	·2552725	·2671717	·2787536	·2900346	·3010300	·3117539	·3222192

TABLE II.—*Logarithms found by adding those in the last Table as indicated.*

	$b=1.$	$b=2.$	$b=3.$	$b=4.$	$b=5.$	$b=6.$	$b=7.$	$b=8.$	$b=9.$	$b=10.$	$b=11.$	$b=12.$	$b=13.$	$b=14.$
1 − log. ϕ_{14}														
1 − log. ϕ_{13}													4·6869838	3·9582556
1 − log. ϕ_{12}												4·2698375	3·5730404	3·1131576
1 − log. ϕ_{11}											3·8441639	3·1906563	2·7601270	2·4441509
1 − log. ϕ_{10}										3·4402369	2·8027712	2·4125051	2·1233049	1·9000829
1 − log. ϕ_{9}									3·0284194	2·4402369	2·0624085	1·8104452	1·6114215	1·4529249
1 − log. ϕ_{8}								2·6191994	2·0741769	1·7412669	1·4981371	1·3333240	1·1964481	1·0849482
1 − log. ϕ_{7}							2·2132554	1·7161094	1·4209664	1·2183882	1·0588044	·9531128	·8606560	·7839182
1 − log. ϕ_{6}						1·8115752	1·3681574	1·1140494	·9438432	·8204482	·7163817	·6520828	·5918106	·5408802
1 − log. ϕ_{5}					1·4156687	1·0334239	·8240894	·6880807	·5916607	·5194182	·4531403	·4179996	·3809572	·3499947
1 − log. ϕ_{4}				1·0280287	·7166987	·5563026	·4561127	·3870507	·3363882	·2975695	·2568456	·2419084	·2212563	·2038667
1 − log. ϕ_{3}			·6532126	·4259687	·3187587	·2552726	·2130747	·1829307	·1602970	·1426675	·1285429	·1169697	·1073129	·0991314
1 − log. ϕ_{2}		·3010300	·1760913	·1249387	·0969100	·0791813	·0669467	·0579920	·0511525	·0457575	·0413927	·0377885	·0347622	·0321846
1 − log. ϕ_{1}	·0000000	·0000000	·0000000	·0000000	·0000000	·0000000	·0000000	·0000000	·0000000	·0000000	·0000000	·0000000	·0000000	·0000000
1 − log. ψ_{1}	·3010300	·1760913	·1249387	·0969100	·0791813	·0669467	·0579920	·0511525	·0457575	·0413927	·0377885	·0347622	·0321846	·0299633
1 − log. ψ_{2}	·7781513	·4771213	·3467874	·2730013	·2253093	·1918854	·1671365	·1480625	·1329077	·1205739	·1103492	·1017090	·0943325	·0879553
1 − log. ψ_{3}	1·3802113	·8750613	·6478174	·5160393	·4294293	·3679766	·3220385	·2763652	·2578464	·2345173	·2150845	·1986191	·1845091	·1722762
1 − log. ψ_{4}	2·0791813	1·3521826	1·0157941	·8170693	·6847018	·5898253	·5183332	·4524564	·4175473	·3806453	·3497831	·3235579	·3010146	·2814207
1 − log. ψ_{5}	2·8573326	1·8962506	1·4417628	1·1692518	·9857318	·8530667	·7524164	·6633098	·6084328	·5567366	·5125104	·4748256	·4423437	·4140463
1 − log. ψ_{6}	3·7024306	2·4983106	1·9188840	1·5671918	1·3281545	1·1540967	1·0212618	·9063478	·8302816	·7608566	·7015666	·6509169	·6071539	·5689483
1 − log. ψ_{7}	4·6055206	3·1515231	2·4417627	2·0065245	1·7083657	1·4898888	1·3222918	1·2793491	1·0801591	·9913055	·9154464	·8504893	·7942405	·7450396
1 − log. ψ_{8}		2·8504931	3·0060341	2·4836457	2·1233391	1·8578655	1·6532851	1·4803791	1·3563655	1·2465780	1·1528073	1·0723381	1·0025164	·9413343
1 − log. ψ_{9}		4·5908558	3·6080940	2·9955291	2·5704971	2·2558055	2·0123071	1·8077380	1·6573955	1·5253316	1·4124446	1·3153762	1·2309957	1·1569341
1 − log. ψ_{10}			4·2449161	3·5395971	3·0476184	2·6817742	2·3976580	2·1599205	1·9819066	1·8263616	1·6932712	1·5786176	1·4787801	1·3910173
1 − log. ψ_{11}				4·1136384	3·5527684	3·1340718	2·8088325	2·5355841	2·3286941	2·1485809	1·9943012	1·8611642	1 7450479	1·6423293
1 − log. ψ_{12}					4·0842473	3·6111930	3·2424881	2·9335241	2·6966709	2·4910036	2·3146363	2·1621942	2·0290445	1·9116747
1 − log. ψ_{13}						4·1117953	3·6984201	3·3526534	3·0848511	2·8527314	2·6534548	2·4809530	2·3300745	2·1969105
1 − log. ψ_{14}							4·1755414	3·7919861	3·4923364	3·2329426	3·0100021	2·8167451	2·6474949	2·4979405
1 − log. ψ_{15}								4·2406239	3·9183051	3·6308826	3·3835827	3·1689277	2·9807095	2·8142105
1 − log. ψ_{16}									4·3620026	4·0458560	3·7735538	3·5369044	3·3291641	3·1452038
1 − log. ψ_{17}											4·1793191	3·9201212	3·6923420	3·4904375
1 − log. ψ_{18}												4·3180622	4·0697603	3·8494595
1 − log. ψ_{19}														4·2218454

In any column corresponding to a particular value of b placed at the top, the numbers from the centre upwards are successively the values of 1 − log. $\phi(b)$, and those successively from the centre downwards are the values of 1 − log. $\psi(b)$.

TABLE II. (*continued*).

	b=15.	b=16.	b=17.	b=18.	b=19.	b=20.	b=25.	b=30.	b=40.	b=50.	b=70.	b=100.	b=200.	b=500.
1 − log. ϕ_{100}														4·616996
1 − log. ϕ_{90}														3·707105
1 − log. ϕ_{80}														2·903012
1 − log. ϕ_{70}														2·202209
1 − log. ϕ_{60}													4·294021	1·602296
1 − log. ϕ_{50}													2·911505	1·100982
1 − log. ϕ_{40}											6·149176	3·9501612	1·817768	·696080
1 − log. ϕ_{30}										4·872150	3·1861803	2·1083913	·994793	·385494
1 − log. ϕ_{25}								4·1556512	4·256354	3·181821	2·1268572	1·4245980	·679801	·264940
1 − log. ϕ_{20}								3·6785299	2·5156797	1·919986	1·3066302	·8847140	·4266829	·1672201
1 − log. ϕ_{19}							4·2275419	3·2428013	2·2358390	1·7123767	1·1691024	·7931990	·3838315	·1503952
1 − log. ϕ_{18}							3·6746999	2·8448603	1·9762017	1·5185567	1·0400077	·7070129	·3423729	·1344722
1 − log. ϕ_{17}						4·5095366	3·1798499	2·4816824	1·7358695	1·3381006	·9191856	·6260910	·3037940	·1194493
1 − log. ϕ_{16}				4·5790489	4·1531142	3·8105666	2·7361524	2·1506891	1·5140207	1·1706095	·8064814	·5503703	·2675818	·1053247
1 − log. ϕ_{15}		4·7411803	4·2066949	3·8008977	3·4764206	3·2085066	2·3382124	1·8496591	1·3099007	1·0157075	·7017461	·4797992	·2337235	·0920964
1 − log. ϕ_{14}	4·3487785	3·8380903	3·4533673	3·1476852	2·8966370	2·6856279	1·9816651	1·5766578	1·1228140	·8730402	·6048361	·4142977	·2022064	·0797627
1 − log. ϕ_{13}	3·4737172	3·1110916	2·8249784	2·5913827	2·3960347	2·2296959	1·6629063	1·3300854	·9521178	·7422719	·5156130	·3538170	·1730180	·0683217
1 − log. ϕ_{12}	2·7747472	2·5090316	2·2934995	2·1142615	1·9623791	1·8317559	1·3789097	1·1082366	·7972158	·6230855	·4339430	·2982997	·1461458	·0577715
1 − log. ϕ_{11}	2·2007159	2·0038816	1·8412019	1·7040870	1·5867155	1·4849684	1·1270977	·9098689	·6575538	·5151801	·3596970	·2476897	·1215776	·0481104
1 − log. ϕ_{10}	1·7235946	1·5779129	1·4558510	1·3519045	1·2622044	1·1839384	·9052490	·7337776	·5326151	·4182701	·2927503	·2019322	·0993012	·0393365
1 − log. ϕ_{9}	1·3256546	1·2188909	1·1284921	1·0508745	·9834508	·9243011	·7114290	·5788756	·4219168	·3320840	·2329821	·1609736	·0793046	·0314480
1 − log. ϕ_{8}	·9946613	·9178609	·8522857	·7956020	·7460899	·7024523	·5439379	·4441770	·3250068	·2563633	·1802758	·1247614	·0615758	·0244431
1 − log. ϕ_{7}	·7216600	·6679834	·6218368	·5817222	·5465175	·5153657	·4012704	·3287835	·2414607	·1908618	·1345183	·0932443	·0461031	·0183200
1 − log. ϕ_{6}	·4998112	·4638634	·4327806	·4056309	·3817073	·3604637	·2820840	·2317734	·1708796	·1353445	·0955997	·0663722	·0328748	·0130769
1 − log. ϕ_{5}	·3237199	·3011361	·2815129	·2643018	·2490817	·2355250	·1851740	·1525921	·1128876	·0895870	·0634151	·0440958	·0218794	·0087121
1 − log. ϕ_{4}	·1890213	·1761973	·1650074	·1551573	·1464194	·1386150	·1094533	·0904441	·0671301	·0533748	·0378610	·0263670	·0131055	·0052238
1 − log. ϕ_{3}	·0921112	·0860207	·0806865	·0759761	·0717858	·0680339	·0539410	·0446866	·0332718	·0265027	·0188380	·0131387	·0065417	·0026102
1 − log. ϕ_{2}	·0299633	·0280287	·0263289	·0248236	·0234811	·0222764	·0177288	·0147233	·0109954	·0087739	·0062489	·0043648	·0021769	·0008695
1 − log. ϕ_{1}	·0000000	·0000000	·0000000	·0000000	·0000000	·0000000	·0000000	·0000000	·0000000	·0000000	·0000000	·0000000	·0000000	·0000000

1 − log. ψ_1	·0280287	·0263289	·0248236	·0234811	·0222764	·0211893	·0170333	·0142404	·0107239	·0086002	·0061603	·0043214	·0021661	·0008677
1 − log. ψ_2	·0823863	·0774814	·0731283	·0692386	·0657421	·0625820	·0504571	·0422691	·0319132	·0256335	·0183948	·0129216	·0064875	·0026014
1 − log. ψ_3	·1615675	·1521150	·1437094	·1361854	·1294112	·1232798	·0996751	·0836617	·0633217	·0509394	·0366197	·0257588	·0129535	·0051994
1 − log. ψ_4	·2642298	·2490250	·2354798	·2232356	·2123854	·2024610	·1641331	·1380193	·1047144	·0843632	·0607534	·0427921	·0215537	·0086599
1 − log. ψ_5	·3891685	·3671243	·3484536	·3296909	·3138430	·2993710	·2433444	·2049660	·1558669	·1257559	·0907167	·0639814	·0322776	·0129813
1 − log. ψ_6	·5352965	·4954270	·4797325	·4546296	·4330294	·4133144	·3367361	·2841472	·2165647	·1749739	·1264323	·0892873	·0451148	·0181618
1 − log. ψ_7	·7016279	·6530348	·6294948	·5972971	·5692491	·5436482	·4439461	·3752276	·2866026	·2318788	·1678250	·1186711	·0600551	·0241998
1 − log. ψ_8	·8872644	·8291260	·7969859	·7569980	·7218593	·6897762	·5645200	·4778899	·3657838	·2963368	·2148216	·1520949	·0770884	·0310935
1 − log. ψ_9	1·0913843	1·0229460	·9815103	·9330893	·8902637	·8511442	·6980589	·5918332	·4539199	·3682188	·2673507	·1895214	·0962047	·0388413
1 − log. ψ_{10}	1·3132330	1·2337994	1·1824252	1·1239748	1·0739081	1·0272355	·8441869	·7167719	·5508299	·4474001	·3253427	·2309141	·1173940	·0474415
1 − log. ψ_{11}	1·5521150	1·4610432	1·3991343	1·3311003	1·2722758	1·2175672	1·0025494	·8524345	·6563401	·5337598	·3887297	·2762371	·1406465	·0568924
1 − log. ψ_{12}	1·8073875	1·7040812	1·6310834	1·5529491	1·4848839	1·4216872	1·1728111	·9985625	·7702834	·6271815	·4574456	·3254551	·1659524	·0671924
1 − log. ψ_{13}	2·0784542	1·9623592	1·8777558	1·7890383	1·7112803	1·6391711	1·3546547	1·1549097	·8924993	·7275520	·5314257	·3785335	·1933020	·0783390
1 − log. ψ_{14}	2·3647609	2·2353605	2·1386686	2·0389158	1·9510396	1·8696200	1·5477793	1·3212411	1·0228331	·8347620	·6106070	·4354384	·2226858	·0903329
1 − log. ψ_{15}	2·6657909	2·5226022	2·4133697	2·3021572	2·2037649	2·1126580	1·7518993	1·4973323	1·1611358	·9487054	·6949279	·4961362	·2540943	·1031701
1 − log. ψ_{16}	2·9810613	2·8236322	2·7014347	2·5783636	2·4690793	2·3679305	1·9777432	1·6829688	1·3072638	1·0692793	·7843284	·5605942	·2875181	·1178498
1 − log. ψ_{17}	3·3101200	3·1380261	3·0024647	2·8671591	2·7466282	2·6351022	2·1920525	1·8779454	1·4610787	1·1963841	·8787497	·6287801	·3229478	·1313703
1 − log. ψ_{18}	3·6525426	3·4653850	3·3160838	3·1681891	3·0360763	2·9138558	2·4275810	2·0820653	1·6224467	1·3299230	·9781344	·7006621	·3603743	·1467301
1 − log. ψ_{19}	4·0079302	3·8053330	3·6419374	3·4811183	3·3371063	3·2038904	2·6730937	2·2951401	1·7912387	1·4698021	1·0824264	·7762091	·3997884	·1629275
1 − log. ψ_{20}		4·1575155	3·9796902	3·8056294	3·6494173	3·5049204	2·9283662	2·5169888	1·9673300	1·6159301	1·1915709	·8553903	·441181	·1799608
1 − log. ψ_{21}			4·3290249	4·1414215	3·9727237	3·8166743								
1 − log. ψ_{22}					4·3067540	4·1388935								
1 − log. ψ_{25}							4·343929	3·7519882	2·953195	2·437287	1·808244	1·304772	·677512	·276761
1 − log. ψ_{30}							5·974835	5·1828759	4·104960	3·402553	2·538658	1·840690	·961994	·395923
1 − log. ψ_{40}										5·730061	4·318592	3·159106	1·671280	·693775
1 − log. ψ_{50}												4·786890	2·561198	1·071943
1 − log. ψ_{60}												6·703453	3·624532	1·529025
1 − log. ψ_{70}													4·854624	2·063595
1 − log. ψ_{80}														2·674295
1 − log. ψ_{90}														3·359811
1 − log. ψ_{100}														4·118878

The numbers above and below ϕ_{20} and ψ_{20} are found with the help of the Table for $\Gamma(x+1)$ in the *Encyclopædia Metropolitana*, on " Probabilities."

TABLE III.—*Giving the Successive Values of* ϕ, ψ, *for any Value of* b.

	$b=1$.	$b=2$.	$b=3$.	$b=4$.	$b=5$.	$b=6$.	$b=7$.	$b=8$.	$b=9$.	$b=10$.	$b=11$.	$b=12$.	$b=13$.	$b=14$.	$b=15$.	$b=16$.	$b=17$.
ϕ_{16}																	
ϕ_{15}																·00002	·00006
ϕ_{14}															·00004	·00015	·00035
ϕ_{13}													·00002	·00011	·00034	·00077	·00150
ϕ_{12}												·00005	·00027	·00077	·00168	·00310	·00509
ϕ_{11}											·00013	·00065	·00174	·00360	·00630	·00911	·01441
ϕ_{10}										·00036	·00128	·00387	·00753	·01259	·01890	·02643	·03501
ϕ_{9}									·00094	·00363	·00866	·01547	·02447	·03524	·04724	·06041	·07439
ϕ_{8}								·00240	·00843	·01814	·03176	·04642	·06361	·08223	·10124	·12082	·14051
ϕ_{7}							·00612	·00192	·03793	·06048	·08734	·11140	·13783	·16446	·18982	·21390	·23887
ϕ_{6}						·01543	·04284	·07690	·11380	·15120	·19214	·22280	·25597	·28752	·31631	·34367	·36916
ϕ_{5}					·03840	·09259	·14993	·20508	·25606	·30240	·35226	·38194	·41595	·44669	·47455	·49988	·52298
ϕ_{4}				·09375	·19200	·27145	·34985	·41016	·46091	·50400	·55355	·57292	·60082	·62536	·64711	·66650	·68390
ϕ_{3}			·22222	·37500	·48000	·55556	·61225	·65625	·69136	·72000	·74380	·76389	·78106	·79592	·80889	·82031	·83045
ϕ_{2}		·50000	·66667	·75000	·80000	·83333	·85714	·87500	·88889	·90000	·90909	·91667	·92307	·92857	·93333	·93750	·94118
ϕ_{1}	1·00000	1·00000	1·00000	1·00000	1·00000	1·00000	1·00000	1·00000	1·00000	1·00000	1·00000	1·00000	1·00000	1·00000	1·00000	1·00000	1·00000
0	1·00000	1·00000	1·00000	1·00000	1·00000	1·00000	1·00000	1·00000	1·00000	1·00000	1·00000	1·00000	1·00000	1·00000	1·00000	1·00000	1·00000

ψ_1	·50000	·66667	·75000	·80000	·83333	·85714	·87500	·88889	·90000	·90909	·91667	·92307	·92857	·93333	·93750	·94118	·94444
ψ_2	·16667	·33333	·45000	·53052	·59524	·64286	·68056	·71111	·73636	·75758	·77570	·79121	·80476	·81667	·82721	·83660	·84503
ψ_3	·04167	·13333	·22453	·30476	·37202	·42857	·47639	·52922	·55227	·58275	·60942	·63297	·65387	·67255	·68934	·70451	·71828
ψ_4	·00833	·04444	·09643	·15238	·20668	·25714	·30316	·35281	·38234	·41625	·44691	·47473	·50002	·52309	·54421	·56360	·58146
ψ_5	·00139	·01270	·03616	·06773	·10334	·14026	·17684	·21712	·24636	·27750	·30725	·33510	·36112	·38544	·40816	·42941	·44828
ψ_6	·00020	·00318	·01205	·02709	·04697	·07013	·09522	·12407	·14782	·17344	·19881	·22340	·24708	·26981	·29154	·31958	·33133
ψ_7	·00002	·00071	·00362	·00985	·01957	·03236	·04761	·05256	·08315	·10202	·12150	·14109	·16060	·17987	·19878	·22231	·23470
ψ_8		·00014	·00099	·00328	·00753	·01587	·02222	·03308	·04401	·05668	·01034	·08466	·09942	·11446	·12964	·14821	·15959
ψ_9		·00003	·00025	·00101	·00269	·00555	·00972	·01557	·02201	·02983	·03869	·04838	·05875	·06967	·08102	·09485	·10435
ψ_{10}			·00006	·00029	·00090	·00208	·00400	·00693	·01043	·01492	·02026	·02639	·03321	·04064	·04861	·05838	·06570
ψ_{11}				·00008	·00028	·00073	·00155	·00291	·00469	·00710	·01013	·01377	·01799	·02276	·02805	·03469	·03989
ψ_{12}					·00008	·00024	·00057	·00117	·00201	·00323	·00485	·00688	·00935	·01227	·01558	·01977	·02338
ψ_{13}						·00008	·00020	·00044	·00082	·00140	·00227	·00330	·00500	·00644	·00834	·01090	·01325
ψ_{14}							·00006	·00016	·00032	·00059	·00098	·00153	·00225	·00318	·00432	·00582	·00727
ψ_{15}								·00006	·00012	·00023	·00041	·00068	·00105	·00153	·00216	·00300	·00386
ψ_{16}									·00004	·00009	·00017	·00029	·00047	·00072	·00121	·00150	·00198
ψ_{71}											·00007	·00012	·00020	·00032	·00050	·00073	·00099
ψ_{18}												·00005	·00009	·00014	·00022	·00034	·00048
ψ_{19}														·00006	·00010	·00016	·00023
ψ_{20}																·00007	·00011

It is sufficiently obvious how this last Table is formed, from the one preceding it. It requires, of course, the extension of this Table to give for the particular values required all the successive values of $\phi_1(b)$, $\phi_2(b)$, &c., $\psi_1(b)$, $\psi_2(b)$, &c., till they become so small as to be safely neglected. All the ψ's below the given percentage being added for the numerator, and the whole sum of the ϕ's and ψ's, + unity, to include the chance of the average number itself being taken for the denominator, the numbers in the final Table are obtained.

TABLE III. (*continued*).

	b=18.	b=19.	b=20.	b=25.	b=30.	b=40.	b=50.	b=70.	b=100.	b=200.	b=500.
ϕ_{100}											·00002
ϕ_{90}											·00019
ϕ_{80}											·00125
ϕ_{70}											·00628
ϕ_{60}										·00005	·02499
ϕ_{50}										·00102	·07925
ϕ_{40}									·00011	·01521	·20133
ϕ_{30}							·00001	·00065	·00779	·10121	·41163
ϕ_{25}						·00006	·00066	·00747	·03762	·20902	·54332
ϕ_{20}					·00021	·00305	·01203	·04936	·13040	·37438	·68042
ϕ_{19}				·00006	·00057	·00581	·01939	·06775	·16099	·41321	·70730
ϕ_{18}				·00021	·00143	·01057	·03030	·09120	·19633	·45460	·73372
ϕ_{17}			·00003	·00066	·00330	·01837	·04590	·12045	·23654	·49683	·75954
ϕ_{16}	·00003	·00007	·00015	·00184	·00707	·03062	·06751	·15614	·28160	·54003	·78465
ϕ_{15}	·00016	·00033	·00062	·00459	·01414	·04899	·09645	·19873	·33128	·58382	·80892
ϕ_{14}	·00071	·00127	·00206	·01043	·02651	·07537	·13396	·24841	·38521	·62776	·83222
ϕ_{13}	·00256	·00402	·00589	·02173	·04676	·11166	·18102	·30506	·44277	·67140	·85443
ϕ_{12}	·00769	·01100	·01473	·04179	·07794	·15951	·23818	·36801	·50315	·71426	·87544
ϕ_{11}	·01977	·02590	·03274	·07463	·12306	·21900	·30537	·43682	·56534	·75583	·89514
ϕ_{10}	·04447	·05468	·06547	·12438	·18460	·29335	·38171	·50962	·62816	·79567	·91341
ϕ_{9}	·08895	·10388	·11904	·19434	·26371	·37852	·46550	·58481	·69028	·83310	·93015
ϕ_{8}	·16010	·17943	·19840	·28580	·35960	·47314	·55416	·66027	·75031	·86781	·94527
ϕ_{7}	·26199	·28411	·30523	·39694	·46905	·57351	·64437	·73364	·80678	·89928	·95869
ϕ_{6}	·39298	·41523	·43605	·52229	·58644	·67472	·73224	·80242	·85828	·92710	·97034
ϕ_{5}	·54412	·56353	·58140	·65287	·70373	·77110	·81360	·86414	·90345	·95087	·98014
ϕ_{4}	·69846	·71381	·72675	·77722	·81200	·85678	·88435	·91651	·94109	·97027	·98804
ϕ_{3}	·83951	·84765	·85500	·88320	·90222	·92625	·94080	·95755	·97020	·98505	·99401
ϕ_{2}	·94444	·94737	·95000	·96000	·96667	·97500	·98000	·98571	·99000	·99500	·99800
ϕ_{1}	1·00000	1·00000	1·00000	1·00000	1·00000	1·00000	1·00000	1·00000	1·00000	1·00000	1·00000

0	1·00000	1·00000	1·00000	1·00000	1·00000	1·00000	1·00000	1·00000	1·00000	1·00000	1·00000
ψ_{1}	·94737	·95000	·95238	·96154	·96774	·97561	·98039	·98592	·99010	·99502	·99800
ψ_{2}	·85263	·85952	·86580	·89031	·90735	·92915	·94268	·95853	·97068	·98517	·99403
ψ_{3}	·73083	·74232	·75287	·79492	·82478	·86433	·88933	·91914	·94241	·97061	·98810
ψ_{4}	·59809	·61322	·62739	·68528	·72775	·78575	·82345	·86945	·90617	·95158	·98026
ψ_{5}	·46807	·48546	·50191	·57102	·62378	·69845	·74859	·81149	·86302	·92837	·97055
ψ_{6}	·35105	·36895	·38609	·46054	·51983	·60734	·66838	·74742	·81417	·90133	·95904
ψ_{7}	·25276	·26962	·28599	·35979	·42148	·51689	·58630	·67948	·76090	·87085	·94580
ψ_{8}	·17499	·17707	·20428	·27257	·33275	·43074	·50543	·60979	·70454	·83736	·93091
ψ_{9}	·11166	·12875	·14088	·20042	·25596	·35163	·42833	·54032	·64637	·80130	·91445
ψ_{10}	·07517	·08435	·09392	·14316	·19196	·28130	·35694	·47278	·58761	·76314	·89652
ψ_{11}	·04665	·05342	·06059	·09942	·14046	·22063	·29258	·40857	·52937	·72336	·87722
ψ_{12}	·02799	·03274	·03787	·06717	·10033	·16971	·23595	·34878	·47266	·68241	·85666
ψ_{13}	·01625	·01944	·02295	·04419	·06000	·12809	·18726	·29415	·41828	·64076	·83495
ψ_{14}	·00914	·01129	·01350	·02833	·04773	·09488	·14630	·24513	·36691	·59884	·81221
ψ_{15}	·00499	·00626	·00772	·01771	·03182	·06900	·11254	·20187	·31905	·55706	·78855
ψ_{16}	·00264	·00339	·00431	·01053	·02075	·04929	·08526	·16431	·27505	·51580	·76234
ψ_{17}	·00136	·00179	·00232	·00643	·01325	·03459	·06362	·13221	·23508	·47539	·73897
ψ_{18}	·00068	·00092	·00122	·00374	·00828	·02385	·04678	·10516	·19922	·43614	·71330
ψ_{19}	·00033	·00046	·00063	·00212	·00507	·01617	·03390	·08271	·16741	·39830	·68718
ψ_{20}	·00016	·00022	·00031	·00118	·00304	·01078	·02421	·06433	·13951	·36209	·66075
ψ_{25}				·00005	·00018	·00111	·00367	·01555	·04957	·21013	·52770
ψ_{30}						·00008	·00040	·00289	·01443	·10914	·40186
ψ_{40}								·00005	·00069	·02132	·20242
ψ_{50}									·00002	·00275	·08473
ψ_{60}										·00024	·02958
ψ_{70}										·00001	·00864
ψ_{80}											·00212
ψ_{90}											·00044
ψ_{100}											·00008

Finally, TABLE IV., *showing the probability on the hypothesis of the preceding investigation of the occurrences in a given period exceeding the average by more than a named percentage.*

Per Cent.	Average Numbers.									
	20.	**25.**	**30.**	**40.**	**50.**	**70.**	**100.**	**200.**	**500.**	**1,000.**
5	·35544	·37062	·38137	·33820	·35417	·33189	·28711	·23407	·19821	·05604
10	·27939	·29981	·25568	·23413	·21553	·18387	·14711	·07755	·02002	·00087
15	·21250	·23659	·20270	·11705	·14156	·10658	·06314	·01766	·00074	·00001 –
20	·15677	·13668	·11963	·09247	·07216	·04483	·02263	·00275	·00001 –	
25	·11218	·09998	·08897	·03551	·04239	·02114	·00675	·00029		
30	·07788	·07144	·04625	·02806	·01726	·00672	·00166	·00002		
40	·03433	·02244	·01482	·00658	·00297	·00063	·00006			
50	·01347	·00919	·00396	·00120	·00037	·00004				

The numbers above the columns denote the numbers that should occur according to the observed average; the numbers on the left, the percentage of deviation in excess of the average. The numbers in a column (say the one headed 50) opposite a certain percentage (say 25), give the probability of the occurrences in a particular case, in which, according to the average observations, they should be 50, exceeding this number by 25 per cent.—that is, in this case, of their being 63 or more. A table such as the above appears to be the best form of representing, at a glance, the relation between the stability of the probable results and the magnitude of the field on which they are looked for.

It is easy to see how the table (IV.) thus finally arrived at, gives the solution of the problem with which we commenced.

Table III. gives a more detailed view of the probabilities of each particular number occurring, and will also itself furnish a very good general idea of what I call the stability of these results.

The case chiefly contemplated in the foregoing pages is that of life assurance; but the columns corresponding to small values of b will give a good idea of the stability to be expected in such a business as marine assurance (for instance).

The somewhat large values of b will express the kind of risks incurred in fire and life businesses, which have not secured a large connexion. The columns for higher values of b explain how the risk arising from fluctuations of luck becomes diminished as the number of transactions increase—becoming, indeed, in the case of very large Offices, an element of much subordinate importance to the fluctuations in the money market, which appear less reducible to law.

Take, for example, the case of an Office which calculates upon losing 70 lives in the course of the year. The odds against the losses in the course of the year *exceeding* this number by 20 per cent.—that is, being greater than 84, are about 24 to 1—that is, a result which we may expect to occur only about once in 25 years.

But, looking at the possible losses in a septennial valuation of assets, we shall have to look at the result of taking 490 (say 500) as the calculated average. I must remark, by the way, that this case does not precisely fulfil the conditions of the problem, because the losses in each year are not entirely independent of one another; but the result of the distinction will simply be, practically to increase slightly the stability to be expected.

In the case now contemplated, the chances are about 5 to 1 against the losses exceeding the average by 5 per cent.—and about 50 to 1 against their exceeding the average by 10 per cent. Again: in the case now contemplated, it is considerably more than 10,000 to 1 against the number exceeding the average by 20 per cent.; and the enormous ratio in which these numbers afterwards diminish, is just the mathematical expression for the certainty of which practice has assured us that the results of such contingent transactions on a sufficiently large scale, will, within certain limits, compensate each other with unerring certainty.

I must remark, that the method by which the general part of the problem is treated in the present article, might have been offered with some hesitation, but for the identity of its result with the limiting case of the problem as more completely analysed in the former article referred to. The former method seems unexceptionable, as it affords a complete analysis of the combinations of events represented by the different numbers, and shows completely what an enormous proportion of the possible combinations of occurrences is represented by the arrangements, which nearly equalize the number of occurences in each year.

I believe the foregoing pages will be of some interest in a practical as well as a theoretical point of view. No doubt, those who are deeply conversant with the practical working of Offices, may acquire a kind of empirical knowledge of the relation between the stability and extent of such business; but, even to the experienced, it is thought that greater precision of ideas may be attained by the expression of this relation in figures; and, moreover, it is something to be assured that the stability thus experienced requires for its explanation no recondite principle of moral or physical laws, but is simply demonstrated by an arithmetical computation of the number of possible events, between which there are no data for assigning preference of expectation.

For the sake of readers less familiar with the principles on which the reasoning with which we started is founded, it may be well to state the result of the present problem in its most elementary form.

Take the case above supposed, where 500 is the average. The proposition is this:—Out of various parts of the country, and from various occupations, &c., a number of people has been taken, of whom, according to the laws of mortality, which (for the present) we suppose known with sufficient accuracy, 500 would die in the course of the year. For every combination of circumstances which would involve the death of a number greater than 10 per cent. of the above, there are 50 which would involve the number being below this limit. For every combination of circumstances involving the death of a number exceeding the 500 by 20 per cent., there are more than 10,000 equally probable combinations of circumstances which will keep the number within that limit. It must be kept in view that the problem assumes—what, of course, is not quite realized—the true ascertainment of the average law. This, no doubt, in the first form of the problem, is implicitly involved along with the oscillations about the average here investigated; but, I think, the assumption already made will be scarcely disputed—namely, that it may be sufficiently ascertained, in order to give the problem of the oscillations here treated a sufficient standing ground of its own. That being admitted, the stability of the results, practically important as it is, is simply the arithmetical consequence, and the law of the oscillations about the average is that expressed by the above tables.

The American Life Underwriters' Convention.

[We drew attention to this important movement in a previous Number,* and now present our readers with the Report of a committee of the Convention on vital statistics. It will be observed from this Report that data have been already collected for the required purpose more extensive than those from which our "Combined Experience" Table has been deduced, and that the records of many Companies are still to be supplied. This we need hardly say is only as it should be, since it seems probable that divisions of the experience must be made on account of differences of climate in so wide a range of country, and since also so many of the Companies are but of recent origination and can afford but slender information as to the lives of persons of advanced age. Great skill and judgment will evidently be required in the general arrangement of the data, to say nothing of that demanded in the ordinary processes of the work. The carrying out of the whole, it appears, has been confided to the committee, and we have reason to believe that no better or wiser course could have been adopted. In a future Number we hope to give the Reports sent in by other committees of the Convention.—Ed. *A. M.*]

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* *See* vol. viii., p. 268.

Report of the Committee on Vital Statistics.—At a Convention of American Life Underwriters, held in this city one year ago, the following preamble and resolutions, offered by Mr. Chickering, of Pittsfield, Mass., were unanimously adopted:—

" Whereas it is the opinion of this Convention (in accordance with the Report already adopted) that it is very desirable that each of the Companies in the United States should unite in contributing towards the determination of the general experience of the rate of mortality among assured lives in the United States—

" Resolved—That Messrs. Homans and Eadie, of New York, and Russell, of Hartford, be appointed a committee to carry into effect the Report on Vital Statistics, adopted by the Convention.

" Resolved—That in conferring with the different Life Companies, and soliciting statements from those not represented in this Convention, the committee assure them that such statements shall be regarded as strictly confidential.

" Resolved—That this committee be instructed to treat all statements furnished by the several Companies as strictly confidential, and to make public only the combined experience ascertained as the result of such statements. That as soon as the necessary blanks can be procured, said committee furnish them to the different Companies, requesting as early replies, and up to as recent dates, as the committee shall see fit.

" Resolved—That the expenses of collecting and publishing said statistics be paid by the Companies jointly, and each Company contributing their experience shall be entitled to one or more copies of the same."

The committee thus appointed have diligently considered the important subjects committed to their care, and have spared no pains to induce all American Life Companies to unite in a project of such practical utility and interest.

In order to carry out the objects thus contemplated, the committee addressed a circular letter to each of the following Companies:—Union Mutual, Portland, Maine; National, Montpelier, Vermont; Massachusetts Hospital, Boston, Mass.; New England Mutual, Boston, Mass.; State Mutual, Worcester, Mass.; Berkshire Mutual, Pittsfield, Mass.; Massachusetts Mutual, Springfield, Mass.; American Temperance, Hartford, Connecticut; Charter Oak, Hartford, Connecticut; Ætna, Hartford, Connecticut; Connecticut Mutual, Hartford, Connecticut; American Mutual, New Haven, Connecticut; Mutual Life, New York; New York, New York; United States, New York; Manhattan, New York; Knickerbocker, New York; New York Life and Trust, New York; Guardian, New York; Washington, New York; Equitable, New York; Home, Brooklyn, N. Y.; Royal, of London and Liverpool, England; International, London, England; Liverpool and London, England; Eagle and Albion, London, England; British Commercial, London, England; Mutual Benefit, Newark, N. J.; Penn Mutual, Philadelphia, Penn.; Pennsylvania, Philadelphia, Penn.; Girard, Philadelphia, Penn.; American, Philadelphia, Penn.; United States, Philadelphia, Penn.; Philadelphia, Philadelphia, Penn.; Baltimore, Baltimore, Md.; Southern Mutual, Columbia, S. C.; Wisconsin Mutual, Milwaukie, Wisconsin; Canada, Hamilton, Canada West; in all, thirty-nine Companies.

To this circular favourable replies were received from the following Companies, who have agreed to contribute from the records of their policies the data requested to a general fund:—Union Mutual, Portland, Maine; National, Montpelier, Vermont; N. E. Mutual, Boston, Mass.; Berkshire, Pittsfield, Mass.; Am. Temperance, Hartford, Connecticut; Ætna, Hartford, Connecticut; Charter Oak, Hartford, Connecticut; Connecticut Mutual, Hartford, Connecticut; New York Life, New York, N. Y.; Manhattan, New York, N. Y.; Knickerbocker, New York, N. Y.; Mutual Life, New York, N. Y.; Washington, New York, N. Y.; Equitable, New York, N.Y.; Guardian, New York, N. Y.; Mutual Benefit, Newark, N. J.; Penn Mutual, Philadelphia, Penn.; Girard Life and Trust, Philadelphia, Penn.; Am. Life and Trust, Philadelphia, Penn.; Southern Mutual, Columbia, S.C.; Canada Life, Canada; Royal, Liverpool and London, England; in all, twenty-two Companies.

The United States, of New York, contemplate the publication of their experience in a separate form.

Completed returns from the following Companies have been received by the committee:—National, Etna, Connecticut Mutual, Royal (of Liverpool and London), Equitable, Girard Life and Trust, American Temperance, Charter Oak, New York Life, Mutual Life of New York, Washington, American Life and Trust, and Southern Mutual; in all, thirteen Companies.

No replies have been received from other Companies, with the exception of one letter from the Secretary of the American Mutual, of New Haven, declining to furnish the *data*, but offering to furnish *results* deduced by themselves, which proposition the committee would respectfully refer to the Convention for action.

The committee have thought that the object contemplated in their appointment would be the better attained if the contributions from the various Companies were to remain unopened until further action by the Convention, when it was hoped that all contributing Companies, including those *not* represented in the last Convention, would be present, and have a voice in the final disposition of the data. The committee have then confined themselves to the duty of inducing as many Companies as possible to contribute their quotas to a general fund, and now place the various contributions at the disposal of the Convention in the same order as when first received.

The committee have thus collected in a simple, concise, yet comprehensive form, the elementary data according to the official record of thirteen Life Companies, and have, in addition, data promised to them from nine other Companies. It is not yet known how many lives and policies, or rather *years of life*, are thus embraced, but the number is undoubtedly larger than that from which was deduced the celebrated "Actuaries'," or "Combined Experience" Table of Mortality.

It now remains for the Convention to decide upon what shall be done with the valuable materials thus gathered, at the cost of so much time and labour.

There is at the present moment a fund of no less than 22,000,000 dollars held in trust and invested by American Life Companies, with annual premiums amounting to more than 7,000,000 dollars, covering policies of about 180,000,000 dollars on the lives of nearly 60,000 American citizens. Is it not time, in a mere pecuniary point of view, that some efforts were made to ascertain the laws governing the duration of life in the localities thus covered? It has been well said that a knowledge of the laws of mortality is the very essence and foundation of the system of life insurance; in fact, a table of mortality may be called the keystone of the arch upon which the vast superstructure of life insurance is based, and upon the accuracy of our tables depend, in a great measure, the stability and safety of the great institution (amounting, in this country, to so many millions of a most sacred fund), in which we are all so much interested.

We have now, Mr. President, from the best of all sources—viz., abstracts from the actual records of the Companies themselves, the preliminary data for constructing tables and deducing sound and reliable information of the greatest practical value. In short, Sir, we have now the opportunity of substituting certainty for uncertainty, facts for mere conjecture, and of placing the whole system of insurance, heretofore greatly dependent upon individual judgment, or at least upon foreign observations, upon the broad foundation of scientific investigation. If the contributing Companies would so arrange their books as that, at stated times—once in five or ten years, for instance—a *census* might be taken, so that we then could correct or corroborate our tables from actual experience, little more could be desired.

In placing the contributions in the hands of the Convention, the committee would recommend that the task of deducing practical tables having special reference to the comparative mortality in the different sections into which our country has been divided, at different ages or epochs of life, both among native and foreign assured, the determination of the extra risk on voyages, &c., and the value of selection, &c., be confided to a proper committee.

Sheppard Homans.
Thomas W. Russell.
New York, May, 1860. John Eadie.

A long discussion seems to have followed the reading of the Report. We have space only for some observations by Mr. Homans, the chairman of the committee, with reference to their *modus operandi*. That gentleman says, "The committee have not consulted together very much on the best plan to be adopted in relation to the matter before the Convention; but it seems to me that the best plan would be, perhaps, to appoint a committee to deduce the results, and give them authority to employ a person to arrange these data in a convenient form. I take it for granted that no person employed in a Life Company could give the time necessary for such a work. I know it would be impossible for me to do it, but it could be done by a clerk at a salary of, say 50 dollars a month, who would arrange the matters in such a way that the committee could at once deduce the information they require, and the different tables that they think requisite, or that the Convention thinks requisite. That, I think, would be the cheapest and the most certain way of arriving at the information sought for. Of course, the tables of mortality could only be deduced by persons who have given some little attention to that subject. There are many persons in this country who would be willing to give their assistance freely—and, indeed, this is not a very small labour—for the sake of helping on the cause of Life Insurance. I have no doubt that any committee which the Convention will appoint will freely give their information and labour."

CORRESPONDENCE.

AUTHORSHIP OF THE TREATISE ON PROBABILITY PUBLISHED BY THE SOCIETY FOR THE DIFFUSION OF USEFUL KNOWLEDGE.

To the Editor of the Assurance Magazine.

DEAR SIR,—In your last Number (p. 143) my friend Sir John Lubbock says, in a note relative to the Useful Knowledge treatise on probability by himself and the late Mr. Drinkwater Bethune, that "if anyone shall pretend that this work was written by De Morgan, I can produce the letter of my lamented friend with which he furnished our manuscript to Mr. Coates." This is a sly joke about the state of things which I shall describe; but to those who are not in the secret there will be something mysterious about it. I am glad of the opportunity of once more setting right a mistake which has now lasted at least fifteen years.

When anything anonymous on mathematics turns up in the Useful Knowledge publications, it was written by me: this is one of the fundamental laws of thought. Bacon says that the mind delights in springing up to general maxims, that it may find rest; and the mind of the cataloguemaker finds rest in the maxim that I am the author of all the anonymous mathematics published by the Useful Knowledge Society. About 1845, a binder, in obedience to the general law, stamped an issue of Lubbock and Drinkwater (Bethune) on *Probability* with the title "De Morgan on Probabilities," in gold letters. In the *Arabian Nights*, when a traveller tells the Sultan some particularly veracious story, it is ordered to be written in letters of gold and deposited in the archives of the kingdom. The story above is as well entitled to the bright alphabet as any which the Princess Schehezerade (or Scheherezade, I forget which) ever told for one day more of life; but our manners, though not wholly averse from lying, properly gilt, do not absolutely require us to perpetuate false titles; I therefore printed one correction of this mistake in my "Arithmetical Books" (1847), and others in other places. But what can a sober black-and-white statement, shut up inside a book, do against a brilliant misstatement on the outside? The only chance is to make the statement long enough to attract attention, and I have spun this letter out accordingly.

Yours truly,

November 8, 1860. A. DE MORGAN.